ON THE BALAZARD-SAIAS CRITERION FOR THE RIEMANN HYPOTHESIS

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1. Introduction

Recently, Balazard and Saias [BS2] have shown that

$$\lim_{N\to\infty}\inf_{D_N}\int_{-\infty}^{\infty}\left|\frac{1-\zeta(\frac12+it)D_N(\frac12+it)}{\frac12+it}\right|^2 dt=0$$

implies the Riemann Hypothesis, where

$$D_N(s):=\sum_{n\le N}\frac{d_n}{n^s}$$

ranges over all Dirichlet polynomials of length N.

It is natural that one may wish to investigate this integral taking for D_N a partial sum of the Dirichlet series for $1/\zeta(s)$,

$$\sum_{n\le N}\frac{\mu(n)}{n^s}.$$

However, this choice has some deficiencies, mainly due to the sharp cutoff of the sum at N, and it is known that this choice does not lead to the desired conclusion.

A better choice is $D_N=M_N$ where

$$M_N(s):=\sum_{n\le N}\frac{\mu(n)\frac{\log(N/n)}{\log N}}{n^s}=\sum_{n\le N}\frac{b_n}{n^s}.$$

M_N has its origins in the works of Selberg and is the mollifier used in Levinson's work on critical zeros of the Riemann zeta-function. Recently, Conrey and Farmer (in preparation)

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have shown that if the Riemann Hypothesis is true and if the zeros of $\zeta(s)$ are separated from each other, in the sense that there is a $\delta > 0$ such that for each zero ρ the derivative of ζ satisfies

$$|\rho|^{1-\delta}|\zeta'(\rho)| \gg 1,$$

then

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\left|\frac{1-\zeta(\frac12+it)M_N(\frac12+it)}{\frac12+it}\right|^2\,dt=0.$$

It is not difficult to deduce by the criterion of Balazard and Saias that the Riemann Hypothesis follows from

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\zeta(\frac12+it)M_N(\frac12+it)}{\frac12+it}\right|^2\,dt=1.$$

(Square out the integrand and use Cauchy's theorem to evaluate the easy terms that arise.)

Proposition 1. *We have*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\zeta(\frac12+it)M_N(\frac12+it)}{\frac12+it}\right|^2\,dt=\int_0^{\infty}\left|\sum_{n\le N}\frac{b_n\{nu\}}{n}\right|^2\frac{du}{u^2}$$

where

$$\{x\}=x-[x]$$

is the fractional part of x.

Proof. The left side is

$$\sum_{h,k\le N}\frac{b_hb_k}{k}F(h/k)$$

where

$$F(x)=\frac{1}{2\pi i}\int_{(\frac12)}\frac{\zeta(s)\zeta(1-s)}{s(1-s)}x^{-s}\,ds;$$

the notation $(\frac12)$ stands for the vertical path from $\frac12-i\infty$ to $\frac12+i\infty$. Now F may be expressed as a convolution

$$F(x)=\int_0^{\infty}f(u)g(x/u)\frac{du}{u}$$

where

$$f(u)=\frac{1}{2\pi i}\int_{(\frac12)}\frac{\zeta(s)}{s}u^{-s}\,ds=\frac{-1}{u}+\left[\frac{1}{u}\right]$$

and

$$g(u) = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \frac{\zeta(1-s)}{1-s} u^{-s} \, ds = \frac{1}{u}(-u + [u]).$$

By a change of variable

$$F(h/k) = \frac{1}{h} \int_0^\infty \{hu\}\{ku\} \frac{du}{u^2},$$

and the proposition follows.

Thus, it is natural to ask about the series

$$W_N(\alpha) = \sum_{n \le N} \frac{\mu(n) \frac{\log(N/n)}{\log N} \{n\alpha\}}{n}. \tag{1}$$

In this paper we show in Theorem 1 that

$$\lim_{N\to\infty} W_N(\alpha) = \frac{-\sin(2\pi\alpha)}{\pi}$$

uniformly for all real α.

We remark that

$$\int_0^\infty \left(\frac{\sin(2\pi u)}{\pi} \right)^2 \frac{du}{u^2} = 1$$

but see Remark 1 after Theorem 2.

This research was carried out while the first author was visiting Macquarie University. He thanks the Department of Mathematics at Macquarie University for its hospitality during a very pleasant visit.

2. Heuristics and statements of theorems

The series in (1) breaks up into $W_N(\alpha) = U_N(\alpha) - \frac{1}{\log N} V_N(\alpha)$ where

$$U_N(\alpha) = \sum_{n \le N} \frac{\mu(n)\{n\alpha\}}{n}$$

and

$$V_N(\alpha) = \sum_{n \le N} \frac{\mu(n)\{n\alpha\} \log n}{n}.$$

To motivate our work we observe that by the prime number theorem,

$$\sum_{n=1}^\infty \frac{\mu(n)}{n} = 0,$$

so that

$$U_N(\alpha) = \sum_{n=1}^{N} \frac{\mu(n)\psi(n\alpha)}{n} + o(1) \tag{2}$$

where the saw-tooth function $\psi(x)$ is defined to be zero at integer arguments and

$$\psi(x) = x - [x] - 1/2 = -\sum_{m=1}^{\infty} \frac{\sin(2\pi m x)}{\pi m}$$

for non-integral x. If we naively insert this series for $\psi(x)$ into the sum in (2) and group terms with $mn = k$ we are led to guess that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\mu(n)\psi(n\alpha)}{n} &= -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{m=1}^{\infty} \frac{\sin(2\pi m n\alpha)}{\pi m} \\ &= -\sum_{k=1}^{\infty} \frac{\sin(2\pi k\alpha)}{\pi k} \sum_{n|k} \mu(k) = -\frac{1}{\pi}\sin(2\pi\alpha). \end{aligned}$$

The series involved are only conditionally convergent so that the interchange of summation is not easily justified.

In [D1] and [D2], Davenport addressed the question of the convergence of $U_N(\alpha)$. In the first paper, he showed that

$$\lim_{N\to\infty} U_N(\alpha) = -\frac{1}{\pi}\sin(2\pi\alpha)$$

for almost all α. In the second paper, after Vinogradov's methods were developed, he showed that the formula is true for all real α and the convergence is uniform. In 1976 S. Segal [S] showed how to derive the formula from a Mellin transform. His method does not seem to show that the convergence is uniform.

A similar argument for

$$V_N^*(\alpha) := \sum_{n\le N} \frac{\mu(n)\log n\psi(n\alpha)}{n}$$

leads one to guess that

$$\lim_{N\to\infty} V_N^*(\alpha) = \sum_{n=1}^{\infty} \frac{\Lambda(n)\sin(2\pi n\alpha)}{\pi n}.$$

Davenport did not address this particular series. Segal's theorem is rather general and shows that the identity above holds in the sense that if either side converges, then so does the other side and to the same value.

It is the goal of this paper to prove

Theorem 1. *Let*

$$W_N(\alpha) = \sum_{n=1}^{N} \frac{\mu(n)\frac{\log(N/n)}{\log N}\{n\alpha\}}{n}.$$

Then,

$$\lim_{N\to\infty} W_N(\alpha) = -\frac{\sin(2\pi\alpha)}{\pi}$$

uniformly for all real α.

In order to do accomplish this goal, we need the following result, which is of independent interest (see Remark 2).

Theorem 2. *The series*

$$T(\alpha) = \sum_{n=1}^{\infty} \frac{\Lambda(n)\sin(2\pi n\alpha)}{\pi n}$$

converges for all real α. The convergence is bounded in the sense that there is an absolute constant $c > 0$ such that the partial sums

$$\left| \sum_{n\le N} \frac{\Lambda(n)\sin(2\pi n\alpha)}{\pi n} \right| \le c$$

for all N and α.

Remark 1. We cannot conclude that the Riemann Hypothesis holds because we cannot show that

$$\frac{1}{u^2}\sum_{n\le N} \frac{\mu(n)\frac{\log(N/n)}{\log N}\{nu\}}{n} \to -\frac{\sin(2\pi u)}{\pi u^2}$$

uniformly. In fact, one can see that if $0 < u < 1/N$ then

$$\sum_{n\le N} \frac{\mu(n)\frac{\log(N/n)}{\log N}\{nu\}}{n} = u\sum_{n\le N}\mu(n)\frac{(\log N/n)}{\log N}$$

so that the integral from 0 to $1/N$ of the square of this expression is just

$$\frac{1}{N}\left|\sum_{n\le N}\mu(n)\frac{\log(N/n)}{\log N}\right|^2.$$

The sum over n has an explicit formula; it is

$$\frac{1}{2\pi i\log N}\int_{(c)} \frac{N^s}{\zeta(s)}\frac{ds}{s^2} = \frac{1}{\log N}\sum_{\rho}\frac{N^\rho}{\zeta'(\rho)\rho^2} + o(1),$$

say, on assuming that the zeros are simple and that $|\zeta'(\rho)\rho| \gg |\rho|^\delta$ for some $\delta > 0$ (the integral is from $c - i\infty$ to $c + i\infty$ where $c > 1$). In this case the series is absolutely convergent and the size of the sum depends on $\sup_\rho |N^\rho|$. If the Riemann Hypothesis is true, this series is bounded uniformly by $N^{1/2}$ from which it follows that

$$\sum_{n \le N} \mu(n) \frac{(\log N/n)}{\log N} \ll \frac{N^{1/2}}{\log N}$$

and so the integral from 1 to $1/N$ is $\ll 1/\log^2 N$. The upshot is that handling the integral over this beginning range clearly depends on the Riemann Hypothesis.

Remark 2. The function $T(\alpha)$ seems to be rather interesting. It appears to be continuous at all irrationals, and to have a jump discontinuity at a/q, with a jump on either side of size $\frac{1}{2}\mu(q)/\phi(q)$ and to satisfy

$$T\left(\tfrac{a}{q}\right) = \lim_{n\to\infty} \tfrac{1}{2}\left(T\left(\tfrac{a}{q} + \tfrac{1}{n}\right) + T\left(\tfrac{a}{q} - \tfrac{1}{n}\right)\right).$$

However, we have not proven these assertions.

3. Preliminaries

In Davenport's paper it is remarked that it is easy to use the theory of L-functions to show that

$$\lim_{N\to\infty} U_N(a/q) = -\frac{1}{\pi}\sin(2\pi a/q)$$

for rational a/q. He does not give the proof. Though it is strictly speaking not needed for what we do, we believe that it is instructive nevertheless. Thus, we will show, using the theory of L-functions,

Proposition 2. *If $(a, q) = 1$, then*

$$\lim_{N\to\infty} U_N(a/q) = \lim_{N\to\infty} \sum_{n\le N} \frac{\mu(n)\{na/q\}}{n} = \frac{-\sin(2\pi a/q)}{\pi}$$

For a Dirichlet character χ modulo q the Dirichlet L-function is defined for $s = \sigma + it$ with $\sigma > 1$ by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

If $q > 1$, then $L(s, \chi)$ can be analytically continued as an entire function. If $q = 1$, then $L(s, \chi) = \zeta(s)$ has a simple pole at $s = 1$ but is analytic everywhere else.

Proposition 3. *If $(a,q)=1$, then*

$$\lim_{N\to\infty}\sum_{n\le N}\frac{\mu(n)\log n\psi(na/q)}{n}$$
$$=\frac{1}{\pi i\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi \text{odd}}}\chi(a)\tau(\overline{\chi})\frac{L'}{L}(1,\chi)+\sum_{p|q}\log p\sum_{k=1}^{\infty}\frac{\sin(2\pi ap^k/q)}{\pi p^k}$$

Proposition 4. *If $(a,q)=1$, then*

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)\sin(2\pi an/q)}{n}=\frac{1}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\text{ odd}}}\frac{\chi(a)\tau(\overline{\chi})}{i}\frac{L'}{L}(1,\chi)+\sum_{p|q}\log p\sum_{k=1}^{\infty}\frac{\sin(2\pi ap^k/q)}{p^k}.$$

Remark It is not difficult to give a finite expression for $\frac{L'}{L}(1,\chi)$, namely

$$\frac{L'}{L}(1,\overline{\chi})=\log 2\pi+\frac{\gamma}{2}+\frac{\sum_{a=1}^{q}\chi(a)\log\Gamma(\frac{a}{q})}{\sum_{a=1}^{q}\chi(a)\frac{a}{q}},$$

where γ is Euler's constant.

We also need

Proposition 5. *There is an absolute constant $c_1>0$ such that the sums $V_N(\alpha)$ satisfy*

$$|V_N(\alpha)|\le c_1$$

for all $N\ge 1$ and all α.

The basic idea of the proofs of Propositions 2 – 4 is to use the fact that $\{na/q\}$ is a periodic function of n with period q. We capture the arithmetic progressions modulo divisors of q by using characters, and eventually we arrive at an expression involving Dirichlet L-functions for odd characters at the special values 0 and 1. We make use of the functional equation for the L-function to arrive at the result.

We can express $L(s,\chi)$ in terms of the Hurwitz zeta-function, defined for $\alpha>0$ and $\sigma>1$ by

$$\zeta(s,\alpha)=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s}.$$

The formula is

$$L(s,\chi)=q^{-s}\sum_{b=1}^{q}\chi(a)\zeta(s,b/q).$$

Since $\zeta(0,b/q)=1/2-b/q$ (see [WW], section 13.21) we have

$$L(0,\chi)=\sum_{b=1}^{q}\chi(b)(1/2-b/q).$$

Lemma 1. *Let χ be a primitive character. Then*

$$L(0,\overline{\chi})L(1,\chi)^{-1} = \begin{cases} \frac{\tau(\overline{\chi})}{\pi i} & \text{if } \chi \text{ is odd} \\ 0 & \text{if } \chi \text{ is even} \end{cases}$$

Proof. If χ is an even primitive character and $q > 1$, then

$$L(0,\chi) = -\frac{1}{q}\sum_{b=1}^{q} b\chi(b) = 0.$$

If $q = 1$, then

$$L(1,\chi)^{-1} = \zeta(1)^{-1} = 0.$$

Thus, the formula is true if χ is even.

If χ is an odd primitive character, then $L(s,\chi)$ satisfies the functional equation (see [D])

$$\pi^{-\frac{1}{2}(2-s)}q^{\frac{1}{2}(2-s)}\Gamma\left(\tfrac{2-s}{2}\right)L(1-s,\overline{\chi}) = \frac{iq^{\frac{1}{2}}}{\tau(\chi)}\pi^{-\frac{1}{2}(s+1)}q^{\frac{1}{2}(s+1)}\Gamma\left(\tfrac{s+1}{2}\right)L(s,\chi)$$

where $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{b=1}^{q}\chi(b)e(b/q)$$

with the usual notation $e(x) = e^{2\pi ix}$. We put $s = 0$ into this formula, and use the facts $\Gamma(1/2) = \pi^{1/2}$ and

$$\tau(\chi)\tau(\overline{\chi}) = \chi(-1)q$$

to obtain the formula in this case.

Lemma 2. *For $(a,q) = 1$ we have*

$$\frac{1}{i\phi(q)}\sum_{\substack{\chi \bmod q \\ \chi \text{odd}}}\chi(a)\tau(\overline{\chi}) = \sin(2\pi a/q).$$

Proof. We have

$$\begin{aligned}
\frac{1}{i\phi(q)}\sum_{\substack{\chi \bmod q \\ \chi \text{odd}}}\chi(a)\tau(\overline{\chi}) &= \frac{1}{2i\phi(q)}\sum_{\chi \bmod q}\left(\chi(a)-\chi(-a)\right)\tau(\overline{\chi}) \\
&= \frac{1}{2i\phi(q)}\sum_{\chi \bmod q}\left(\chi(a)-\chi(-a)\right)\sum_{b=1}^{q}\overline{\chi}(b)e(b/q) \\
&= \frac{1}{2i\phi(q)}\sum_{b=1}^{q}e(b/q)\sum_{\chi \bmod q}\overline{\chi}(b)\left(\chi(a)-\chi(-a)\right) \\
&= \frac{1}{2i}\left(e(a/q)-e(-a/q)\right) = \sin(2\pi a/q).
\end{aligned}$$

4. Proofs

Proof of Proposition 2. Let

$$U_N^*(\alpha) = \sum_{n \le N} \frac{\mu(n)\psi(n\alpha)}{n}.$$

By (2), this is equal to $U_N(a/q) + o(1)$. Then

$$U_N^*(a/q) = \sum_{b=1}^{q} \psi(ab/q) \sum_{\substack{n \le N \\ n \equiv b \bmod q}} \frac{\mu(n)}{n}.$$

We let $g = (n, q)$. Then

$$U_N^*(a/q) = \sum_{g|q} \frac{\mu(g)}{g} \sum_{\substack{b=1 \\ (b,q)=1}}^{q/g} \psi\left(\frac{ab}{q/g}\right) \sum_{\substack{n \le N/g \\ n \equiv b \bmod (q/g) \\ (n,g)=1}} \frac{\mu(n)}{n}.$$

Since $(b, q/g) = 1$ we can express the congruence condition in the sum over n by using characters modulo q/g. Thus, the sum over n is

$$\frac{1}{\phi(q/g)} \sum_{\chi \bmod (q/g)} \overline{\chi}(b) \sum_{\substack{n \le N/g \\ (n,q/g)=1}} \frac{\mu(n)\chi(n)}{n}.$$

We change variables in the sum over b and replace b by $b\overline{a}$ where $a\overline{a} \equiv 1 \pmod{q/g}$. We have

$$U_N^*(a/q) = \sum_{g|q} \frac{\mu(g)}{g} \frac{1}{\phi(q/g)} \sum_{\chi \bmod (q/g)} \chi(a) \sum_{b=1}^{q/g} \overline{\chi}(b)\psi\left(\frac{b}{q/g}\right) \sum_{\substack{n \le N/g \\ (n,q/g)=1}} \frac{\mu(n)\chi(n)}{n}.$$

The sum over b is $-L(0, \overline{\chi})$. Thus,

$$U_N^*(a/q) = -\sum_{g|q} \frac{\mu(g)}{g} \frac{1}{\phi(q/g)} \sum_{\chi \bmod (q/g)} \chi(a) L(0, \overline{\chi}) \sum_{\substack{n \le N/g \\ (n,q/g)=1}} \frac{\mu(n)\chi(n)}{n}. \tag{3}$$

Recall that $L(0, \chi) = 0$ if χ is a non-principal character to an even modulus. So, we can restrict the sum over χ above to characters that are either odd or principal.

The sum over n in (3) is

$$\sum_{\substack{n\le N/g\\(n,q/g)=1}}\frac{\mu(n)\chi(n)}{n}=\frac{1}{2\pi i}\int_{(c)}\frac{\prod_{p|g}\left(1-\chi(p)/p^s\right)(N/g)^s}{L(s+1,\chi)}\frac{ds}{s}$$
$$\sim L(1,\chi)^{-1}\prod_{p|g}\left(1-\frac{\chi(p)}{p}\right)^{-1}$$

by the prime number theorem for arithmetic progressions.

Thus, we now have

$$U_N^*(a/q)=-\sum_{g|q}\frac{\mu(g)}{g}\frac{1}{\phi(q/g)}\sum_{\substack{\chi \bmod \frac{q}{g}\\ \chi \text{odd}}}\chi(a)L(0,\overline{\chi})L(1,\chi)^{-1}+E_N(a/q)$$

where $E_N(a/q)\to 0$ as $N\to\infty$ for fixed a and q.

To further simplify the main term we use Lemma 1. But first we have to reduce to primitive characters. If χ mod q is induced by χ_1 mod q_1 where χ_1 is primitive, then

$$L(s,\chi)=L(s,\chi_1)\prod_{p|(q/q_1)}\left(1-\frac{\chi_1(p)}{p^s}\right).$$

Thus, we can write our main term as

$$-\sum_{g|q}\frac{\mu(g)}{g}\frac{1}{\phi(q/g)}\sum_{r|(q/g)}\sideset{}{^*}\sum_{\substack{\chi \bmod r\\ \chi \text{odd}}}\chi(a)L(0,\overline{\chi})\prod_{p|\frac{q}{gr}}(1-\overline{\chi}(p))$$
$$L(1,\chi)^{-1}\prod_{p|\frac{q}{rg}}\left(1-\frac{\chi(p)}{p}\right)^{-1}\prod_{\substack{p|g\\ p\nmid\frac{q}{g}}}\left(1-\frac{\chi(p)}{p}\right)^{-1}$$

where the * denotes that the sum is for primitive characters. We combine two of the products and use Lemma 1 to rewrite the above as

$$-\frac{1}{\pi i}\sum_{g|q}\frac{\mu(g)}{g}\frac{1}{\phi(q/g)}\sum_{r|(q/g)}\sideset{}{^*}\sum_{\substack{\chi \bmod r\\ \chi \text{odd}}}\chi(a)\tau(\overline{\chi})\prod_{p|\frac{q}{gr}}(1-\overline{\chi}(p))\prod_{p|\frac{q}{r}}\left(1-\frac{\chi(p)}{p}\right)^{-1}.$$

We exchange the orders of summation of g and r and expand one of the products to see that the above is

$$-\frac{1}{\pi i}\sum_{r|q}\sideset{}{^*}\sum_{\substack{\chi \bmod r\\ \chi \text{odd}}}\prod_{p|\frac{q}{r}}\left(1-\frac{\chi(p)}{p}\right)^{-1}\sum_{d|\frac{q}{r}}\mu(d)\overline{\chi}(d)\sum_{g|\frac{q}{rd}}\frac{\mu(g)}{g\phi(q/g)}.$$

The sum over g is

$$\begin{cases} \frac{\mu^2(\frac{q}{rd})rd}{q\phi(q)} & \text{if } (rd, q/rd) = 1 \\ 0 & \text{if } (rd, q/rd) > 1 \end{cases}$$

Thus, the sum over d is

$$\sum_{\substack{d|\frac{q}{r} \\ (rd,q/rd)=1}} \mu(d)d\overline{\chi}(d)\mu^2\left(\frac{q}{rd}\right).$$

If $(r, q/r) > 1$, then this sum is 0 because if $p \mid r$ and $p \mid q/r$, then $p \mid d$ (since otherwise $p \mid \frac{q}{rd}$), but then $\chi(d) = 0$ since χ is a character modulo r. Moreover, the sum is 0 if q/r is not squarefree: for if $p^2 \mid \frac{q}{r}$, then $p^2 \mid d$ implies $\mu^2(d) = 0$, $p \,||\, d$ implies $(d, q/rd) > 1$, and $p \nmid d$ implies $\mu^2(q/rd) = 0$.

Thus, our main term can be rewritten as

$$-\frac{1}{\pi i q\phi(q)} \sum_{\substack{r|q \\ (r,q/r)=1}} r\mu^2\left(\frac{q}{r}\right) \sideset{}{^*}\sum_{\substack{\chi \bmod r \\ \chi \text{ odd}}} \chi(a)\tau(\overline{\chi}) \prod_{p|\frac{q}{r}} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \prod_{p|\frac{q}{r}} (1 - p\overline{\chi}(p)).$$

Now

$$\frac{1 - p\overline{\chi}(p)}{1 - \frac{\chi(p)}{p}} = \frac{p\chi(p) - p^2}{p\chi(p) - \chi(p)^2} = -p\overline{\chi}(p)$$

so that the products over p reduce to

$$\frac{q}{r}\mu\left(\frac{q}{r}\right)\overline{\chi}\left(\frac{q}{r}\right).$$

Thus, our main term can now be written as

$$-\frac{1}{\pi i\phi(q)} \sum_{\substack{r|q \\ (r,q/r)=1}} \sideset{}{^*}\sum_{\substack{\chi \bmod r \\ \chi \text{odd}}} \chi(a)\mu\left(\frac{q}{r}\right)\overline{\chi}\left(\frac{q}{r}\right)\tau(\overline{\chi}).$$

Now if χ mod q is induced by χ_1 mod r then $\tau(\chi) = 0$ if $(r, q/r) > 1$ or if $\mu(q/r) = 0$. If $(r, q/r) = 1$ and q/r is squarefree, then

$$\tau(\chi) = \mu\left(\frac{q}{r}\right)\chi_1\left(\frac{q}{r}\right)\tau(\chi_1).$$

Thus, the above expression for our main term simplifies to

$$-\frac{1}{\pi i\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{odd}}} \chi(a)\tau(\overline{\chi}).$$

Now

$$\sum_{\substack{\chi \bmod q \\ \chi \text{odd}}} \chi(a)\tau(\overline{\chi}) = \frac{1}{2}\sum_{\chi \bmod q} (\chi(a) - \chi(-a))\tau(\overline{\chi}).$$

Also,

$$\sum_{\chi \bmod q} \chi(a)\tau(\overline{\chi}) = \sum_{\chi \bmod q} \sum_{b=1}^{q} \overline{\chi}(b)e(b/q) = \phi(q)e(a/q).$$

Thus, the main term reduces to

$$\frac{-\sin(2\pi a/q)}{\pi}$$

as desired.

Proof of Proposition 3. We reduce this Proposition to several instances of Proposition 2. To do this, we write

$$\begin{aligned} W_N(a/q) &= \sum_{n\le N} \frac{\mu(n)\log\left(\frac{n}{(n,q)}\right)\psi\left(naq\right)}{n} + \sum_{n\le N} \frac{\mu(n)\log(n,q)\psi\left(naq\right)}{n} \\ &= \Sigma_1 + \Sigma_2 \end{aligned}$$

say. We handle Σ_1 much as in the proof of Proposition 2. We split the range of summation into arithmetic progressions $b \bmod q$ and split further according to the greatest common divisor $g = (b,q) = (n,q)$. Thus, we arrive at

$$\Sigma_1 = -\sum_{g|q} \frac{\mu(g)}{g}\frac{1}{\phi(q/g)} \sum_{\chi \bmod \frac{q}{g}} \chi(a)L(0,\overline{\chi}) \sum_{\substack{n\le N/g \\ (n,q/g)=1}} \frac{\mu(n)\chi(n)\log n}{n}.$$

Now

$$\begin{aligned} \sum_{\substack{n=1 \\ (n,g)=1}}^{\infty} \frac{\mu(n)\chi(n)\log n}{n} &= \frac{d}{ds}L(s,\chi\chi_{0,g})^{-1}\bigg|_{s=1} \\ &= L(1,\chi)^{-1}\prod_{p|g}\left(1-\frac{\chi(p)}{p}\right)^{-1}\frac{L'}{L}(1,\chi\chi_{0,g}), \end{aligned}$$

where $\chi_{0,g}$ is the principal character modulo g. We can replace the sum over n with this expression and have exactly the same error term $E_N(q)$ as in Proposition 2.

We reduce to primitive characters and use Lemma 1, much as before. The main term of Σ_1 is then

$$-\frac{1}{\pi i}\sum_{r|q}\sideset{}{^*}\sum_{\substack{\chi \bmod r\\ \chi \text{odd}}}\prod_{p|\frac{q}{r}}\left(1-\frac{\chi(p)}{p}\right)^{-1}\sum_{d|\frac{q}{r}}\mu(d)\overline{\chi}(d)\sum_{g|\frac{q}{rd}}\frac{\mu(g)}{g\phi\left(\frac{q}{g}\right)}\frac{L'}{L}(1,\chi\chi_{0,g})$$

This term can now be treated exactly as in the proof of Proposition 2. It leads to a contribution of

$$\frac{1}{\pi i}\sum_{\chi \bmod q}\tau(\overline{\chi})\chi(a)\frac{L'}{L}(1,\chi).$$

To treat Σ_2 we use the formula

$$\log n=\sum_{s|n}\Lambda(s).$$

Thus,

$$\begin{aligned}\Sigma_2&=\sum_{n\le N}\frac{\mu(n)\log(n,q)\psi\left(naq\right)}{n}\\&=\sum_{n\le N}\frac{\mu(n)\psi\left(naq\right)}{n}\sum_{\substack{s|q\\ s|n}}\Lambda(s)\\&=\sum_{s|q}\Lambda(s)\sum_{n\le N/s}\frac{\mu(sn)\psi\left(snaq\right)}{sn}.\end{aligned}$$

Clearly, s must be a prime divisor of q. We change s to p and have

$$\Sigma_2=-\sum_{p|q}\log p\sum_{\substack{n\le N/p\\ p\nmid n}}\frac{\mu(n)\psi\left(napq\right)}{pn}.$$

Now, for any positive integer k let

$$r(k)=\sum_{\substack{n\le x\\ p\nmid n}}\frac{\mu(n)}{np^k}\psi\left(\frac{anp^k}{q}\right).$$

Then,

$$\begin{aligned} r(k) &= \sum_{n\le x} \frac{\mu(n)}{np^k}\psi\left(\frac{anp^k}{q}\right) - \sum_{\substack{n\le x\\ p\mid n}} \frac{\mu(n)}{np^k}\psi\left(\frac{anp^k}{q}\right) \\ &= -\frac{1}{\pi}\frac{\sin(2\pi a p^k/q)}{p^k} + o\left(\frac{1}{p^k}\right) + \sum_{\substack{n\le x\\ p\nmid n}} \frac{\mu(n)}{np^{k+1}}\psi\left(\frac{anp^{k+1}}{q}\right) \\ &= -\frac{1}{\pi}\frac{\sin(2\pi a p^k/q)}{p^k} + o(1) + r(k+1). \end{aligned}$$

If we apply this relation repeatedly, we end up with

$$\Sigma_2 = \sum_{p\mid q} \log p \sum_{k=1}^{\infty} \frac{\sin(2\pi a p^k/q)}{\pi p^k} + o(1).$$

Thus, we have proved Proposition 3.

Proof of Proposition 4. We have

$$\sum_{n=1}^{N} \frac{\Lambda(n)\sin(2\pi an/q)}{n} = \sum_{\substack{n\le N\\ (n,q)=1}} \frac{\Lambda(n)\sin(2\pi an/q)}{n} + \sum_{\substack{n\le N\\ (n,q)>1}} \frac{\Lambda(n)\sin(2\pi an/q)}{n} \tag{4}$$

and

$$\begin{aligned} \sum_{\substack{n\le N\\ (n,q)=1}} \frac{\Lambda(n)\sin(2\pi an/q)}{n} &= \frac{1}{i\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi \text{ odd}}} \chi(a)\tau(\overline{\chi}) \sum_{n\le N} \Lambda(n)\chi(n) \\ &= \frac{1}{i\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi \text{ odd}}} \chi(a)\tau(\overline{\chi}) \frac{L'}{L}(1,\chi) + o(1). \end{aligned}$$

To evaluate the second sum on the right side of (4) we observe that since Λ is supported on prime powers, it must be the case that (n,q) is a power of a prime p, or else the sum is 0. Thus, we can group the terms according to primes p dividing q. For a given p dividing q the n for which $p \mid (n,q)$ and $\Lambda(n)\neq 0$ are just $n = p^k$ for some $k \ge 1$. Therefore, the second sum is

$$\sum_{p\mid q} \log p \sum_{p^k\le N} \frac{\sin(2\pi a p^k)}{p^k} \sim \sum_{p\mid q} \log p \sum_{k=1}^{\infty} \frac{\sin(2\pi a p^k)}{p^k}.$$

To prove Proposition 5 we use the ideas of Davenport [D1] and [D2] . First, we prove

Lemma 3. *We have*

$$\Bigg|\sum_{\substack{n\le N\\ q|n}} \frac{\mu(n)\log n}{n}\Bigg| \ll \begin{cases} \frac{1}{\phi(q)} & \text{if } q \ll \log^h N \\ \frac{\log N}{q} & \text{if } q \ge \log^h N \end{cases}$$

Proof.

To prove this, note that the sum is

$$\begin{aligned} &\frac{1}{q}\Bigg|\sum_{\substack{n\le \frac{N}{q}\\ (q,n)=1}} \frac{\mu(n)\log nq}{n}\Bigg| \\ &\le \frac{1}{q}\Bigg|\sum_{\substack{n\le \frac{N}{q}\\ (q,n)=1}} \frac{\mu(n)\log q}{n}\Bigg| + \frac{1}{q}\Bigg|\sum_{\substack{n\le N\\ (q,n)=1}} \frac{\mu(n)\log n}{n}\Bigg| \end{aligned}$$

The first term is $O\big((\log q)/q\big)$ for all q by [D1] Lemma 1 and is $O\big((\log N)^{-h}\big)$ by Lemma 12 of [D2] for $q \le \log^h N$. So it suffices to bound

$$\sum_{\substack{n\le x\\ (q,n)=1}} \frac{\mu(n)\log n}{n}.$$

Note that

$$\sum_{\substack{d|n\\ (d,q)=1}} \mu(d)\log d = -\Lambda\left(\frac{n}{n_q}\right)$$

where n_q is that part of n which is coprime to q, i.e., $n_q = \prod_{p^k\|n, p\nmid q} p^k$. The n for which $\Lambda(n/n_q) \ne 0$ are those of the form $n = dm$ where $d \in q^\infty$ and $\Lambda(m) \ne 0$ (where q^∞ is the set of all integers all of whose prime factors divide q). Thus,

$$\begin{aligned} \sum_{n\le x}\sum_{\substack{d|n\\ (d,q)=1}} \mu(d)\log d &= -\sum_{d\in q^\infty}\sum_{n\le \frac{x}{d}} \Lambda(n) \\ &\ll \sum_{d\in q^\infty} \frac{x}{d} \ll x\prod_{p|q}\left(1+\frac{1}{p}+\frac{1}{p^2}+\dots\right) \\ &= x\prod_{p|q}(1+1/p) \ll x\prod_{p\le q}(1+1/p) \ll x\log q. \end{aligned}$$

Therefore,

$$\left| \sum_{n \le x} \sum_{\substack{d|n \\ (d,q)=1}} \mu(d) \log d \right| \ll x \log q.$$

But the left side of this inequality is

$$\begin{aligned} \left| \sum_{\substack{d \le x \\ (d,q)=1}} \mu(d) \log d \left[\frac{x}{d} \right] \right| &= x \left| \sum_{\substack{d \le x \\ (d,q)=1}} \frac{\mu(d) \log d}{d} \right| + O(x \log x) \\ &\ll x \log qx. \end{aligned}$$

For $q \le \log^h N$,

$$\sum_{\substack{n \le x \\ (n,q)=1}} \frac{\mu(n) \log n}{n} = \frac{1}{2\pi i} \int_{(c)} \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\mu(n) \log n}{n^s} \frac{x^s}{s} \, ds.$$

The series under the integral sign is

$$-\frac{d}{ds} \left(\zeta(s)^{-1} \prod_{p|q} \left(1 - \frac{1}{p^s} \right)^{-1} \right),$$

which, by standard arguments, is

$$-\prod_{p|q} \left(1 - \frac{1}{p} \right)^{-1} + O\left((\log x)^{-h} \right) = -\frac{q}{\phi(q)} + O\left((\log x)^{-h} \right).$$

Lemma 4. *Let*

$$V_N^*(\alpha) = \sum_{n \le N} \frac{\mu(n) \log n \psi(n\alpha)}{n}.$$

Then for all N, α_1, α_2,

$$|V_N^*(\alpha_1) - V_N^*(\alpha_2)| \ll N \log N |\alpha_1 - \alpha_2| + 1.$$

Proof. The proof follows Lemma 2 of [D1] as well as Lemmas 12 and 13 of [D2]. We have that $V_N^*(\alpha)$ is continuous and differentiable, with derivative

$$\sum_{n \le N} \mu(n) \log n \ll N \log N$$

except at rationals a/q with $q \le N$ where it has a jump discontinuity of size

$$-\sum_{\substack{n\le N\\ q|n}} \frac{\mu(n)\log n}{n}.$$

Thus,

$$V_N^*(\alpha) - V_N^*(\beta) \ll (\alpha-\beta) N\log N + \left| \sum_{\alpha\le\frac{a}{q}\le\beta} \sum_{\substack{n\le N\\ q|n}} \frac{\mu(n)\log n}{n}\right|.$$

Now we use the estimates of Lemma 3 for the inner sum and the arguments of Lemma 2 of [D1] and Lemma 13 of [D2] to complete the proof.

Proof of Proposition 5. Here we follow the proofs of Lemma 14 and Theorem 2 of [D2]. Let

$$\begin{aligned} R_N(\alpha) &= V_N^*(\alpha) - T(\alpha)\\ &= \sum_{n>N} \frac{\mu(n)\log n\psi(n\alpha)}{n}. \end{aligned}$$

Then

$$\int_{\alpha_1}^{\alpha_2} R_N(\alpha)\ d\alpha = \sum_{n>N} \frac{\mu(n)\log n\psi_2(n\alpha_2)}{n} - \sum_{n>N} \frac{\mu(n)\log n\psi_2(n\alpha_1)}{n}$$

where

$$\psi_2(t) = \frac{1}{2\pi^2}\sum_{m=1}^{\infty} \frac{\cos 2\pi m t}{m^2} = \int_0^t \psi(u)\ du + \frac{1}{12}.$$

Thus,

$$\begin{aligned} \sum_{n>N} \frac{\mu(n)\log n\psi_2(n\alpha)}{n} &= \frac{1}{2\pi^2}\sum_{n>N}\frac{\mu(n)\log n}{n^2}\sum_{m=1}^{\infty}\frac{\cos 2\pi m n\alpha}{m^2}\\ &= \frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{n>N}\frac{\mu(n)\log n\cos 2\pi m n\alpha}{n^2}\\ &\ll N^{-1}(\log N)^{-h} \end{aligned}$$

by Theorem 1 of [D2] and partial summation. Next,

$$\begin{aligned} (\alpha_1-\alpha_2)R_N(\alpha_1) &= \int_{\alpha_1}^{\alpha_2} R_N(\alpha_1)\ d\alpha\\ &= \int_{\alpha_1}^{\alpha_2} R_N(\alpha)\ d\alpha + \int_{\alpha_1}^{\alpha_2} (R_N(\alpha_1) - R_N(\alpha))\ d\alpha. \end{aligned}$$

Therefore,

$$|R_N(\alpha_1)| \le \frac{1}{\alpha_1-\alpha_2}\frac{1}{N\log^h N} + \max_{\alpha_1\le\alpha,\beta\le\alpha_2} |R_N(\alpha)-R_N(\beta)|\,.$$

Now

$$R_N(\alpha)-R_N(\beta) = V_N^*(\alpha)-V_N^*(\beta)+T(\alpha)-T(\beta) \ll 1+|V_N^*(\alpha)-V_N^*(\beta)|$$

by Theorem 2 of [D2]. Take

$$\alpha_1-\alpha_2 = \frac{1}{N\log^h N}$$

and use Lemma 4 to obtain the result.

Proof of Theorem 2. Let

$$S_u(\alpha) = \sum_{n\le u}\Lambda(n)\sin(2\pi n\alpha).$$

Then

$$\sum_{n\le N}\frac{\Lambda(n)\sin(2\pi n\alpha)}{n} = \frac{S_N(\alpha)}{N} + \int_2^N \frac{S_u(\alpha)}{u^2}\,du. \tag{5}$$

Note that

$$|S_N(\alpha)| \le \sum_{n\le N}\Lambda(n) \ll N$$

so that the first term on the right side of (5) is uniformly bounded. Now let $H>10$ be fixed. Define

$$\tau = \tau(u) = \frac{u}{\log^H u}$$

for $u\ge 2$. Let $q\le\tau$ be such that

$$\left|\alpha-\frac{a}{q}\right| \le \frac{1}{q\tau}$$

for some a. Note that for each u there is a unique such q. We split the u with $2\le u\le N$ into two sets $R_1(N)$ and $R_2(N)$ according to the size of q. If $q\le\log^H u$ then $u\in R_1(N)$, and if $\log^H u\le q\le\tau(u)$, then $u\in R_2(N)$. We will show that

$$\int_{R_j(N)}\frac{S_u(\alpha)}{u^2}\,du$$

is uniformly bounded and has a limit as $N\to\infty$ for $j=1$ and 2.

Suppose $u \in R_2$. Then, by the theorem of section 25 of [D],

$$\begin{aligned} S_u(\alpha) &\ll \left(\frac{u}{q} + u^{4/5} + (uq)^{1/2} \right) \log^4 u \\ &\ll \frac{u}{(\log u)^{\frac{H}{2}-4}}. \end{aligned}$$

Therefore,

$$\int_{R_2(N)} \frac{S_u(\alpha)}{u^2}\, du \ll \int_2^N \frac{du}{u \log^{1+\delta} u} \ll 1$$

uniformly for all N. The integral over $R_2 = \lim_{N\to\infty} R_2(N)$ is absolutely convergent.

Now suppose that $u \in R_1(N)$. Write

$$\alpha = \frac{a}{q} + \beta.$$

Then by section 26 of [D],

$$S_u(\alpha) = \Im \frac{\mu(q)}{\phi(q)} \sum_{n \le u} e(n\beta) + O\left(u \exp(-C\sqrt{\log u}) \right)$$

for an absolute constant $C > 0$, where $\Im z$ is the imaginary part of z. Clearly, the integral over R_2 of the O-term is uniformly bounded and converges absolutely.

Now

$$\Im \sum_{n \le u} e(n\beta) = \sum_{n=1}^{[u]} \sin(2\pi n\beta) = \frac{\sin\left(\frac{([u]+1)\beta}{2}\right) \sin\left(\frac{[u]\beta}{2}\right)}{\sin\beta}.$$

Thus, for any particular q the integral over R_2 of the contribution from the main term above is bounded by

$$\frac{1}{\phi(q)} \int_2^\infty \left| \frac{\sin\left(\frac{([u]+1)\beta}{2}\right) \sin\left(\frac{[u]\beta}{2}\right)}{\sin\beta} \right| \frac{du}{u^2}. \tag{6}$$

Observe that

$$|\sin([u]\beta) - \sin(u\beta)| \le |\beta|$$

and

$$\frac{|\beta|}{|\sin\beta|} \ll 1$$

so that the expression in (6) is

$$\frac{1}{\phi(q)}\left(\int_2^\infty \left|\frac{\sin^2\left(\frac{u\beta}{2}\right)}{\sin\beta}\right|\frac{du}{u^2}+O(1)\right).$$

Let $v=u\beta$ to see that the above is bounded by

$$\frac{1}{\phi(q)}\left(\int_2^\infty \left|\frac{\sin^2\left(\frac{u}{2}\right)}{u^2}\right|\,du+O(1)\right)\ll\frac{1}{\phi(q)}.$$

All of the q which appear in the above proof are denominators of convergents of the continued fraction of α. It is easy to see that if the convergents of α are p_m/q_m then

$$\sup_\alpha\sum_{n=1}^\infty\frac{1}{\phi(q_m)}\ll 1.$$

Thus, the contribution of this part is uniformly bounded and converges.

Thus, we have completed the proof that the partial sums $V^*(N)$ are uniformly bounded.

It only remains to observe that $\lim_{N\to\infty}S_N(\alpha)/N=0$ for all fixed α to complete the proof of convergence. If α is rational then convergence of $T(\alpha)$ follows from Proposition 4. If α is irrational, then we argue again according to whether $N\in R_1(N)$ or $N\in R_2(N)$. In the first case, the relevant $q\to\infty$, and the second case is clear. Thus, we have convergence in all cases.

Proof of Theorem 1. It follows from [D2] that

$$U_N(\alpha)\to\frac{\sin(2\pi\alpha)}{-\pi}$$

uniformly. Thus, it suffices to show that

$$\frac{1}{\log N}V_N(\alpha)\to 0$$

uniformly. Hence, it suffices to show that $V_N(\alpha)$ is uniformly bounded. Proposition 5 shows that $V_N^*(\alpha)$ is uniformly bounded. If α is irrational then

$$\begin{aligned}V_N(\alpha)&=V_N^*(\alpha)-\frac{1}{2}\sum_{n\le N}\frac{\mu(n)\log n}{n}\\&=V_N^*(\alpha)-1+o(1).\end{aligned}$$

If $\alpha=a/q$ is rational, then

$$V_N(\alpha)=V_N^*(\alpha)-\frac{1}{2}\sum_{n\le N}\frac{\mu(n)\log n}{n}+\frac{1}{2}\sum_{\substack{n\le N\\ q|n}}\frac{\mu(n)\log n}{n}.$$

The last term is uniformly bounded by Lemma 3. Thus, $V_N(\alpha)$ is uniformly bounded and the Theorem follows.

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